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Test for a Specified Signal When the Noise Covariance Matrix Is Unknown*

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In the univariate case it is well known that the one sided t test is uniformly most powerful for the null hypothesis against all one sided alternatives. Such a property does not easily extend to the multivariate case. In this paper, a test derived for the hypothesis that the mean of a vector random variable is zero against specified alternatives, when the covariance matrix is unknown. This test depends on the given alternatives and is more powerful than Hotelling's T^2. The results are derived both for real and complex vector observations and under normal and spherical distributions. The properties of the proposed tests are investigated in detail when a single alternative is specified. © 1987 Academic Press, Inc.

1. INTRODUCTION

In a previous paper [2], the authors discussed the problem of discriminating a specified signal from noise when the covariance matrix of noise was unknown but an estimate was available. In this paper, we consider the analogous problem of testing whether a received message X is

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pure noise against the alternative that it contains a specified signal. We assume that an independent estimate S of the noise covariance matrix is available. First, we discuss the case when X and S are real and have multivariate normal and Wishart distributions in order to facilitate comparison of our procedure with existing methods, and then extend the results to the complex case and spherical distributions.

In the real case, the problem may be formulated as follows. Let $X \sim N_p(\mu, a^{-1}\Sigma)$, i.e., as p-variate normal with mean vector μ and covariance matrix $a^{-1}\Sigma$ where a is a known scalar, and $S \sim W_p(f, \Sigma)$, i.e., as $p \times p$-variate Wishart on f degrees of freedom and covariance matrix Σ. The main problem we consider is that of testing the hypothesis

$$H_0: \mu = 0 \qquad \text{vs} \qquad H_1: \mu = \delta \text{ (specified)} \tag{1.1}$$

based on observed (X, S) when Σ is unknown and X and S are independently distributed. We note that when $p = 1$, the one sided t test of the null hypothesis (1.1)

$$t = \frac{\sqrt{a}\, x}{\sqrt{s/f}} > c \qquad \text{if} \quad \delta > 0 \text{ (or } < c \text{ if } \delta < 0) \tag{1.2}$$

is uniformly most powerful for all $\delta > 0$ (or $\delta < 0$) in the class of similar region tests with respect to the unknown variance σ^2 of X. When $p > 1$, Hotelling's T^2, which is a multivariate analogue of the two sided univariate t test, is generally used to test H_0: $\mu = 0$, and it is known that it provides a uniformly most powerful test for all alternative values of μ in the class of invariant tests (see [3, pp. 299–300]). We show that there exists an alternative test which is more powerful than Hotelling's T^2 for the specified alternative δ of μ. We describe the exact properties of the proposed test.

The same results hold in more general situations where X and S have complex p-variate normal and complex $p \times p$-variate Wishart distributions, and also where X and S have a spherical distribution.

2. Test for the Null Hypothesis (1.1)

Let C be a $p \times (p-1)$ matrix of rank $p-1$ such that $\delta' C = 0$ and consider the transformation

$$Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} \delta' \\ C' \end{pmatrix} X, \qquad V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} = \begin{pmatrix} \delta' \\ C' \end{pmatrix} S(\delta : C). \tag{2.1}$$

Then

$$Y \sim N_p\left(\begin{pmatrix}0\\0\end{pmatrix}, a^{-1}\Sigma_*\right) \quad \text{under} \quad H_0$$

$$\sim N_p\left(\begin{pmatrix}\delta'\delta\\0\end{pmatrix}, a^{-1}\Sigma_*\right) \quad \text{under} \quad H_1$$

$$V \sim W_p(f, \Sigma_*), \Sigma_* = \begin{pmatrix}\delta'\\C'\end{pmatrix} \Sigma(\delta; C) = \begin{pmatrix}\Sigma_{11} & \Sigma_{12}\\ \Sigma_{21} & \Sigma_{22}\end{pmatrix} \tag{2.2}$$

and the problem of testing the hypothesis (1.1) is equivalent to the testing

$$H_0: E(Y_1) = 0 \qquad \text{vs} \qquad H_1: E(Y_1) = \delta'\delta > 0$$

given that $E(Y_2) = 0$. Such a problem was considered in Rao [4].

From (2.1) and (2.2), the conditional distribution of Y_1 given Y_2 is

$$Y_1 \sim N_1(\alpha + \beta' Y_2, a^{-1}\sigma_{1.2}^2) \tag{2.3}$$

with $\alpha = 0$ under H_0 and $\alpha = \delta'\delta$ under H_1, and those of $b = V_{22}^{-1} V_{21}$ and $V_{1.2} = V_{11} - V_{12} V_{22}^{-1} V_{21}$ given V_{22} are

$$\begin{aligned} b &\sim N_{p-1}(\beta, \sigma_{1.2}^2 V_{22}^{-1}) \\ V_{1.2} &\sim \chi^2(f - p + 1, \sigma_{1.2}^2), \end{aligned} \tag{2.4}$$

where $\beta = \Sigma_{22}^{-1}\Sigma_{21}$ and $\sigma_{1.2}^2 = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$. Further, the conditional distributions (2.3) and (2.4) of Y_1, b and $V_{1.2}$ are all independent. In such a case, the problem of testing a linear hypothesis on α and β can be considered within the framework of a univariate linear model. For testing a hypothesis on α we consider its estimate

$$\hat{\alpha} = Y_1 - b' Y_2 \sim N_1(\alpha, \sigma_{1.2}^2(a^{-1} + Y_2' V_{22}^{-1} Y_2)) \tag{2.5}$$

when $Y_2' V_{22}^{-1} Y_2$ is fixed, and the independent statistic providing an estimate of $\sigma_{1.2}^2$,

$$V_{1.2} \sim W_1(f - p + 1, \sigma_{1.2}^2) \qquad \text{or} \qquad \chi^2(f - p + 1, \sigma_{1.2}^2). \tag{2.6}$$

From (2.5) and (2.6), it follows that the most powerful similar region test, with respect to the unknown $\sigma_{1.2}^2$, for the hypothesis $\alpha = 0$ against any $\alpha > 0$ is $t \geqslant c$, where

$$t = (f - p + 1)^{1/2} \hat{\alpha} / [(a^{-1} + Y_2' V_{22}^{11} Y_2) V_{1.2}]^{1/2} \tag{2.7}$$

has the t distribution on $f - p + 1$ degrees of freedom, and c is the critical value of t for a chosen level of significance. In our case the alternative is $\alpha = \delta'\delta > 0$. Hence the appropriate test is as given in (2.7).

The statistic t in (2.7) is defined in terms of transformed variables Y and V. This was done to express our problem in a canonical form to derive the test criterion and state its properties. We can, however, express t in (2.7) explicitly in terms of original variables X and S by using the identity

$$S^{-1} = \frac{S^{-1}\delta\delta' S^{-1}}{\delta' S^{-1}\delta} + C(C'SC)^{-1}C', \tag{2.8}$$

where C is as defined in (2.1). Then, from (2.1) and (2.8) we have

$$\begin{aligned} Y_2' V_{22}^{-1} Y_2 &= X'C(C'SC)^{-1}C'X \\ &= X'S^{-1}X - \frac{(\delta' S^{-1}X)^2}{\delta' S^{-1}\delta}, \\ \hat{\alpha} &= Y_1 - Y_2' b \\ &= X'\delta - X'C(C'SC)^{-1}C'S\delta \\ &= (X'S^{-1}\delta)\,\delta'\delta/\delta' S^{-1}\delta, \\ V_{1.2} &= V_{11} - V_{12}V_{22}^{-1}V_{21} = (\delta'\delta)^2/\delta' S^{-1}\delta. \end{aligned} \tag{2.9}$$

Then t in (2.7) can be expressed in terms of X and S as

$$t = \frac{a^{1/2}(f-p+1)^{1/2}\,\delta' S^{-1}X}{[(1+aX'S^{-1}X)\,\delta' S^{-1}\delta - a(\delta' S^{-1}X)^2]^{1/2}}. \tag{2.10}$$

Thus the appropriate test for the null hypothesis (1.1) is a one-sided t-test on $(f-p+1)$ degrees of freedom with t as defined in (2.10), which explicitly involves the specified alternative mean vector δ.

3. Power of the t-Test (2.10)

From (2.3) and (2.4), the conditional distribution of $\hat{\alpha}/(a^{-1} + Y_2' V_{22}^{-1} Y_2)^{1/2}$ given Y_2 and V is

$$N_1(\alpha/(a^{-1} + Y_2' V_{22}^{-1} Y_2)^{1/2}, \sigma_{1.2}^2) \tag{3.1}$$

and is independent of $V_{1.2}$. Hence the conditional distribution of t defined in (2.7) or (2.10) given $Y_2' V_{22}^{-1} Y_2$ is that of a noncentral Student's t with $(f-p+1)$ degrees of freedom and the noncentral parameter

$$\frac{\alpha^2}{(a^{-1} + Y_2' V_{22}^{-1} Y_2)\,\sigma_{1.2}^2} = \frac{a\delta' \Sigma^{-1}\delta}{1 + aY_2' V_{22}^{-1} Y_2} = \frac{\gamma}{1 + aY_2' V_{22}^{-1} Y_2}. \tag{3.2}$$

Since

$$z = \frac{1}{1 + aY_2' V_{22}^{-1} Y_2} \sim B\left(\frac{f-p+2}{2}, \frac{p-1}{2}\right), \tag{3.3}$$

i.e., beta distribution, the noncentral probability density of t is given by

$$\begin{aligned}
&\frac{\Gamma\left(\frac{f+1}{2}\right)}{(f-p+1)^{1/2}\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{f-p+1}{2}\right)\Gamma\left(\frac{p-1}{2}\right)} \sum_{j=0}^{\infty} \frac{\Gamma\left(\frac{f-p+2+j}{2}\right)}{\Gamma\left(\frac{f-p+2}{2}\right)} \frac{(2\gamma)^{j/2}}{j!} \\
&\quad \times \left(\frac{t}{\sqrt{f-p+1}}\right)^j \left(1 + \frac{t^2}{f-p+1}\right)^{-(f-p+2+j)/2} \\
&\quad \times \int_0^1 e^{-\gamma z/2} z^{(f-p+j)/2} (1-z)^{(p-3)/2}\, dz.
\end{aligned} \tag{3.4}$$

The probability that $t \geqslant t_\alpha$ can be computed by using the above density function. By making the transformation

$$u_\alpha = (f-p+1)^{1/2} t_\alpha / (1 + \overline{f-p+1}\, t_\alpha^2)^{1/2} \tag{3.5}$$

$P(t \geqslant t_\alpha)$ can be written as

$$\sum_0^\infty \frac{(\gamma/2)^{j/2}}{\Gamma\left(\frac{j+2}{2}\right)} \int_0^1 \frac{e^{-\gamma y/2} y^{(f-p+j)/2} (1-y)^{(p-3)/2}}{B\left(\frac{f-p+2}{2}, \frac{p-1}{2}\right)} dy \int_{u_\alpha}^\infty \frac{u^j (1-u^2)^{(f-p-1)/2}}{B\left(\frac{j+1}{2}, \frac{f-p+1}{2}\right)} du \tag{3.6}$$

using the formula

$$j!\, \Gamma\left(\frac{1}{2}\right) = 2^j \Gamma\left(\frac{j+1}{2}\right) \Gamma\left(\frac{j+2}{2}\right).$$

Thus, the power function of the t-test depends only on $\gamma = a\delta' \Sigma^{-1} \delta$.

An alternative to the proposed t-test is the Hotelling's T^2-test for H_0: $\mu = 0$

$$F = \frac{f-p+1}{p} T^2 = \frac{f-p+1}{p} aX'S^{-1}X \tag{3.7}$$

which has F distribution on p and $(f-p+1)$ degrees of freedom. The test criterion (3.7) does not explicitly depend on δ but its power function involves the same parameter $\gamma = a\delta'\Sigma^{-1}\delta$. We show that the power of the test based on Hotelling's T^2 is smaller than that of the proposed t-test at $\mu = \delta$.

For this purpose, let us consider the joint density of $\hat{\alpha}$, $V_{1.2}$, and $T_1^2 = aY_2'V_{22}^{-1}Y_2$,

$$p(\hat{\alpha}, V_{1.2}, T_1^2) = \text{const}\, h_1(T_1^2)\, V_{1.2}^{(f-p-1)/2} \exp\left[-\frac{V_{1.2}}{2\sigma_{1.2}^2} - \frac{(\hat{\alpha}-\alpha)^2}{2\sigma_{1.2}^2(1+T_1^2)}\right],$$

where

$$h_1(T_1^2) = \frac{1}{B\left(\dfrac{p-1}{2}, \dfrac{f-p+2}{2}\right)} \frac{(T_1^2)^{(p-3)/2}}{(1+T_1^2)^{(f+1)/2}}.$$

Now as $aX'S^{-1}X = T^2 = T_1^2 + (\hat{\alpha}^2/V_{1.2})$ and hence the joint density of T^2 and T_1^2 is

$$\begin{aligned} p_1(T_1^2, T^2/\gamma) &= h(T_1^2)(1+T_1^2)^{-1} \exp[-\gamma/2(1+T_1^2)] \\ &\quad \times \sum_0^\infty \left(\frac{\gamma}{2(1+T_1^2)}\right)^j \frac{U^{j+1/2-1}}{j!\, B\left(j+\dfrac{1}{2}, \dfrac{f-p+1}{2}\right)(1+U)^{j+(f-p+2)/2}}, \end{aligned} \tag{3.8}$$

where $U = (T^2 - T_1^2)/(1+T_1^2)$, the statistic introduced by Rao [5, p. 554] to test for additional information. Now for testing $H_0: \gamma = 0$ against $H_1: \gamma \neq 0$, the Neyman–Pearson test based on T^2 and T_1^2 is

$$\frac{p_1(T, T_1^2 \mid \gamma)}{p_1(T, T_1^2 \mid \gamma = 0)} > \lambda. \tag{3.9}$$

The left-hand side of (3.9) is a monotone likelihood ratio (see Srivastava and Khatri [7]), and the inequality of (3.9) is equivalent to $U >$ constant. Thus the optimal test based on T^2 and T_1^2 is

$$U = \frac{T^2 - T_1^2}{1 + T_1^2} > \text{const}, \tag{3.10}$$

where $T^2 = aX'S^{-1}X$ and $T_1^2 = aX'S^{-1}X - a(\delta'S^{-1}X)^2/\delta'S^{-1}\delta$. Substituting these values in (3.9) we find $U = (f-p+1)^{-1}t^2$, where t is as in (2.10). Thus the test based on T^2 is less powerful than that based on t^2 or t for the alternative $\mu = \delta$.

4. Some Auxiliary Tests

The t-test (2.10) provides optimum power for detecting a specified alternative to the null hypothesis, which may be known a priori. However, in practice we may have to envisage the possibility of a different unknown signal being transmitted. We note that the transformed variable Y_2 in (2.1) has the distribution

$$Y_2 \sim N_{p-1}(0, a^{-1}\Sigma_{22}) \tag{4.1}$$

when noise or the specified signal δ is transmitted. Since

$$V_{22} \sim W_{p-1}(f, \Sigma_{22}) \tag{4.2}$$

the hypothesis $E(Y_2)=0$, which implies that only signals of the kind $c\delta$ are transmitted, can be tested by using Hotelling's T^2

$$\frac{a(f-p+2)}{p-1} Y_2' V_{22}^{-1} Y_2 = \frac{a(f-p+2)}{p-1}\left(X'S^{-1}X - \frac{(\delta' S^{-1}X)^2}{\delta' S^{-1}\delta}\right) \tag{4.3}$$

which has F distribution on $(p-1)$ and $(f-p+2)$ degrees of freedom when $E(Y_2)=0$. A high value for (4.3) would indicate the presence of an unknown signal.

The efficiency of the test (2.10) will also depend on the magnitude of the unknown regression parameter $\beta = \Sigma_{22}^{-1}\Sigma_{21}$ defined in (2.3) and (2.4). A test for the hypothesis $\beta=0$ derived from the linear model (2.3) and (2.4) is

$$\begin{aligned} F &= \frac{f-p+1}{p-1}\frac{b'V_{22}b}{V_{1.2}} = \frac{f-p+1}{p-1}\left[\delta' S\delta - \frac{(\delta'\delta)^2}{\delta' S^{-1}\delta}\right]\frac{\delta' S^{-1}\delta}{(\delta'\delta)^2} \\ &= \frac{f-p+1}{p-1}\left[\frac{(\delta' S\delta)(\delta' S^{-1}\delta)}{(\delta'\delta)^2} - 1\right] \end{aligned} \tag{4.4}$$

which has F distribution on $(p-1)$ and $(f-p+1)$ degrees of freedom. If the value of F is small, then a simpler test for the null hypothesis (1.1) is

$$t = \frac{\sqrt{a}\,\delta' X}{(\delta' S\delta/f)^{1/2}} \tag{4.5}$$

which has t distribution on f degrees of freedom. Indeed when $\beta=0$ or close to zero, the t-test (4.5) is more powerful than the t-test (2.10).

Remark. We note that when $\beta=0$, the covariance matrix Σ has the representation

$$\Sigma = \theta\delta\delta' + \phi I, \tag{4.6}$$

where θ and ϕ are arbitrary, and the test (4.4) is indeed a test of the hypothesis on the structure (4.6) of Σ. When (4.6) holds, it is seen that the vector Y_2 is distributed independently of Y_1 and has no information on the parameters of the distribution of Y_1. In such a case, the $t-$test (4.5) based on Y_1 alone is efficient and has more power than the t-test (2.10). There may be other structures of Σ which would imply that certain components of Y_2 are independent of the other components of Y_2 and Y_1. In such a case, we may omit these components of Y_2 in constructing the t-test of (2.10).

5. Extensions to Other Distributions

5.1. *Spherical Distribution*

Some of the results derived in the previous sections hold for more general distributions of X and S. We shall demonstrate this, for instance, when the joint density of X and S is

$$\frac{|S|^{(f-p-1)/2}}{a^{p/2}\Pi^{p/2}\Gamma_p\left(\frac{f}{2}\right)|\Sigma|^{(f+1)/2}} h(\operatorname{tr}\Sigma^{-1}S + a(X-\mu)'\Sigma^{-1}(X-\mu)), \quad (5.1)$$

where

$$\Gamma_p\left(\frac{f}{2}\right) = \Pi^{p(p-1)/4} \prod_{i=1}^{p} \Gamma\left(\frac{f-i+1}{2}\right).$$

Using the transformations made in (2.1), the joint density of b, $V_{1.2}$, Y_1, Y_2, and V_{22} under $H_0\colon \mu = 0$ is

$$\frac{V_{1.2}^{(f-p-1)/2}|V_{22}|^{(f-p+1)/2}}{a^{p/2}\Pi^{p/2}\Gamma_p(f/2)\,\sigma_{1.2}^{(f+1)}|\Sigma_{22}|^{(f+1)/2}}$$

$$\times h\left(\frac{V_{1.2} + (b-\beta)'V_{22}(b-\beta) + a(Y_1 - \beta' Y_2)^2}{\sigma_{1.2}^2} + \operatorname{tr}\Sigma_{22}^{-1}V_{22} + aY_2'\Sigma_{22}^{-1}Y_2\right). \quad (5.2)$$

We can write

$$\begin{aligned}(b-\beta)'V_{22}(b-\beta) &+ a(Y_1-\beta' Y_2)^2 \\ &= (b-\beta+\beta_1)'(V_{22}+aY_2Y_2')(b-\beta+\beta_1) + \frac{a(Y_1 - b'Y_2)^2}{1+aY_2'V_{22}^{-1}Y_2}, \quad (5.3)\end{aligned}$$

where $\beta_1 = a(Y_1 - b'Y_2)(V_{22} + aY_2Y_2')^{-1}Y_2$. Hence the joint density of $b_1 = b - \beta + \beta_1$, $V_{1.2}$, $Y_{(1)} = (Y_1 - b'Y_2)/(1 + aY_2'V_{22}^{-1}Y_2)^{1/2}$, $V_{(22)} = V_{22} + aY_2Y_2'$, and Y_2 is

$$\frac{V_{1.2}^{(f-p-1)/2}|V_{(22)} - aY_2Y_2'|^{(f-p+1)/2}(1 + aY_2'V_{22}^{-1}Y_2)^{1/2}}{a^{(p-1)/2}\Pi^{p/2}\Gamma_p(f/2)\,\sigma_{1.2}^{(f+1)}|\Sigma_{22}|^{(f+1)/2}}$$

$$\times h\left(\frac{V_{1.2} + Y_{(1)}^2 + b_1'V_{(22)}b_1}{\sigma_{1.2}^2} + \operatorname{tr}\Sigma_{22}^{-1}V_{(22)}\right). \tag{5.4}$$

From this the following easily follow:

(1) The t defined in (2.7 or 2.10) has Student's distribution on $(f-p+1)$ degrees of freedom.

(2) The distribution of $1/(1 + aY_2'V_{22}^{-1}Y_2)$ is beta as in (3.3) whether $\mu = 0$ or δ, so that the test (4.3) of the hypothesis that $E(Y_2) = 0$ remains valid.

(3) The test for $\beta = 0$ is the same as in (4.4) based on F distribution with $(p-1)$ and $(f-p+1)$ degrees of freedom.

Thus the tests of nulll hypotheses considered in Sections 2, 3, and 4 of the paper are robust with respect to the wider class of distributions (5.1) involving an arbitrary function h of the variables X and S. But the same results do not hold for the non-null distributions. (For further details on null robustness of certain multivariate tests, reference may be made to Khatri [1] and Sinha and Drygas [6]).

5.2. *Complex Case*

Let X be a complex random vector and S be a $p \times p$ Hermitian positive definite random matrix with the joint density function

$$\frac{|S|^{f-p}}{a^p\Pi^p\tilde{\Gamma}_p(f)|\Sigma|^{f+1}}h(\operatorname{tr}\Sigma^{-1}S + a(X-\mu)^*\Sigma^{-1}(X-\mu)), \tag{5.5}$$

where Σ is Hermitian positive definite, X^* denotes the conjugate transpose of X and μ denotes the complex mean vector.

All the tests developed for the real case are valid for the complex case also, and can be obtained by replacing the transpose (′) by the conjugate transpose (*). For example, the test for H_0: $\mu = 0$ against H: $\mu = \delta$ (given) is

$$t = \frac{\{2(f-p+1)\}^{1/2}\,\delta^*S^{-1}X}{\{(a^{-1} + X^*S^{-1}X)\,\delta^*S^{-1}\delta - \delta^*S^{-1}XX^*S^{-1}\delta\}^{1/2}} \tag{5.6}$$

and H_0 is rejected if (real part of t) $\geqslant t_\alpha$, where t_α is a constant depending on the size of the test. The real part of t is distributed as Student's t with $2(f-p+1)$ degrees of freedom.

Further, the test whether the signal δ is transmitted is given by the statistic

$$\frac{a(f-p+2)}{p-1}\left(X^*S^{-1}X-\frac{\delta^*S^{-1}XX^*S^{-1}\delta}{\delta^*S^{-1}\delta}\right) \tag{5.7}$$

which has F distribution (under H_0) with $2(p-1)$ and $2(f-p+2)$ degrees of freedom and it does not depend on h.

The test for H_0: $\delta^*\Sigma C=0$ (or $\Sigma_{12}\Sigma_{22}^{-1}=0$) is based on the statistic

$$\frac{f-p+1}{p-1}\left[\frac{(\delta^*S\delta)(\delta^*S^{-1}\delta)}{(\delta^*\delta)^2}-1\right] \tag{5.8}$$

which has F distribution (under H_0) with $2(p-1)$ and $2(f-p+1)$ degrees of freedom and it does not depend on h.

The power function of the above test procedures do depend on the structure of h. The power of the t test (5.6) can be obtained in the same way as was done in the real case in Section 2.

6. Tests for a Wider Null Hypothesis

6.1. *Real Case*

Let us consider the distribution of X and S as in (5.1) with an arbitrary h function and the null hypothesis

$$H_0: \mu=0 \qquad \text{vs} \qquad H_1: \mu \in R(G), \tag{6.1}$$

where $R(G)$ is the range space of a given $p\times r$ matrix G of rank r.

First, we make a transformation similar to (2.1), choosing a $p\times(p-r)$ matrix C of rank $(p-r)$ such that $G'C=0$,

$$Y=\begin{pmatrix}Y_1\\Y_2\end{pmatrix}=\begin{pmatrix}G'\\C'\end{pmatrix}X, \qquad V=\begin{pmatrix}V_{11} & V_{12}\\V_{21} & V_{22}\end{pmatrix}=\begin{pmatrix}G'\\C'\end{pmatrix}S(G:C). \tag{6.2}$$

Then $E(Y_2)=0$ under both H_0 and H_1 mentioned in (6.1). The problem (6.1) can be reformulated as

$$H_0: E(Y_1)=0 \qquad \text{vs} \qquad H_1: E(Y_1)=G'Gv \tag{6.3}$$

given $E(Y_2)=0$, where v is arbitrary. (Note that Y_1 in this case is an r vector and the alternative $G'Gv$ is unspecified.)

Following the arguments as in Section 2, and Rao [4; 5, p.554] the appropriate statistic is found to be

$$U=\frac{f-p+1}{r}\frac{a(Y_1-b'Y_2)'V_{1.2}^{-1}(Y_1-b'Y_2)}{(1+aY_2'V_{22}^{-1}Y_2)} \tag{6.4}$$

which has F distribution on r and $(f-p+1)$ degrees of freedom under H_0, where

$$b=V_{22}^{-1}V_{21} \qquad \text{and} \qquad V_{1.2}=V_{11}-V_{12}b.$$

In terms of the original variables, the statistic (6.4) can be written as

$$U=\frac{f-p+1}{r}\frac{aX'S^{-1}G(G'S^{-1}G)^{-1}G'S^{-1}X}{1+a[X'S^{-1}X-X'S^{-1}G(G'S^{-1}G)^{-1}G'S^{-1}X]}. \tag{6.5}$$

The null distribution of U is thus independent of the h function in the joint density (5.1) of X and S. But its non-null distribution may depend on h. In the case when X has normal and S has Wishart distribution, the power function of U can be computed as in (3.6)

$$\begin{aligned}P(U>U_\alpha)=\sum_{j=0}^{\infty}\left(\frac{\gamma}{2}\right)^j\frac{1}{j!}\int_{r_\alpha}^{\infty}\frac{u^{(r/2)+j-1}(1-u)^{(f-p-1)/2}}{B(r/2+j,(f-p+1)/2)}\,du\\ \times\int_0^1\frac{e^{-\gamma y/2}y^{[(f-p+r+1)/2]+j-1}(1-y)^{(p-r-2)/2}}{B((f-p+r+1)/2,(p-r)/2)}\,dy,\end{aligned} \tag{6.6}$$

where

$$r_\alpha=\frac{rU_\alpha}{f-p+1}\Big/\left(1+\frac{rU_\alpha}{f-p+1}\right)$$
$$\gamma=av'(G'\Sigma^{-1}G)\,v=a(\mu'\Sigma^{-1}\mu).$$

Similarly, the test for H_0: $\mu\in R(G)$ versus H_1: μ is arbitrary can be based on Hotelling's T^2 (see [5, pp. 564–565]). The derived F-statistic is

$$F=\frac{f-p+r+1}{p-r}a(X'S^{-1}X-X'S^{-1}G(G'S^{-1}G)^{-1}G'S^{-1}X) \tag{6.7}$$

on $(p-r)$ and $(f-p+r+1)$ degrees of freedom. This holds for the general model considered in (5.1).

To test the hypothesis H_0: $G'\Sigma C=0$ (or Σ is of the form $\Sigma=dI+GDG'$ where $d>0$ and D is any positive definite matrix), we have to choose a suitable function of the eigenvalues of

$$b'V_{22}bV_{1.2}^{-1}=(G'SG)^{-1}G'G(G'S^{-1}G)^{-1}G'G-I_r. \tag{6.8}$$

The null distribution of (6.8) is independent of h, and hence the large sample approximation of the distribution of any chosen function of eigenvalues based on the normality assumption can be used.

Finally for the complex case, all the above results remain valid with appropriate changes:

(i) x', G', b', Y_2', etc., are to be changed to x^*, G^*, b^*, Y_2^*, etc.

(ii) The degrees of freedom are doubled.

(iii) $\gamma/2$ is to be changed to $\gamma = a\nu^*(G^*\Sigma^{-1}G)\,\nu = a\mu^*\Sigma^{-1}\mu$.

(iv) The power function of U under complex normality assumption can be computed using the formula

$$P(U > U_\alpha) = \sum_{j=0}^{\infty} \frac{\gamma^j}{j!} \int_{r_\alpha}^{\infty} \frac{u^{r+j-1}(1-u)^{f-p}}{B(r+j, f-p+1)}\, du \times \int_0^1 \frac{e^{-\gamma y} y^{f-p+r+j}(1-y)^{p-r-1}}{B(f-p+r+1), p-r)}\, dy. \tag{6.9}$$

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